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# Lattice formulation of $\mathcal{N} = 4$ super Yang-Mills theory

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Abstract: We construct a lattice action for $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions which is local, gauge invariant, free of spectrum doubling and possesses a single exact supersymmetry. Our construction starts from the observation that the fermions of the continuum theory can be mapped into the component fields of a single real anticommuting Kähler-Dirac field. The original supersymmetry algebra then implies the existence of a nilpotent scalar supercharge $Q$ and a corresponding set of bosonic superpartners. Using this field content we write down a $Q$-exact action and show that, with an appropriate change of variables, it reduces to a well-known *twist* of $\mathcal{N} = 4$ super Yang-Mills theory due to Marcus. Using the discretization prescription developed earlier [1] we are able to translate this geometrical action to the lattice.

Keywords: Topological Field Theories, Lattice Gauge Field Theories, Extended Supersymmetry.



 

## Contents





## 1. Introduction

Attempts to formulate lattice supersymmetric theories have a long history (see [2] and the recent reviews by Feo and Kaplan [3, 4]). Recently there has been renewed interest in this problem stemming from the realization that in certain classes of theory it may be possible to preserve at least some of the supersymmetry *exactly* at finite lattice spacing. It is hoped that this residual supersymmetry may protect the lattice theory from dangerous SUSY-violating radiative corrections. In the case of extended supersymmetry two approaches have been followed;[1] in the first the lattice theory is constructed by orbifolding a certain supersymmetric matrix model [5, 6]. The second approach relies on reformulating the supersymmetric theory in terms of a new set of variables — the *twisted* fields. In this procedure a scalar nilpotent supercharge is exposed and it is the algebra of this charge that one may hope to preserve under discretization [9]. This approach was initially used to construct lattice formulations of a variety of low dimensional theories without gauge symmetry [10 − 12]. A possible generalization to lattice gauge theories was given by Sugino [13]. However, Sugino's models in four dimensions suffer from the presence of additional states which do not appear to decouple in the limit of vanishing lattice spacing.

In this paper we introduce a new discretization of the $\mathcal{N} = 4$ twisted Yang-Mills action which is a generalization of the procedure used earlier to construct a lattice theory of $\mathcal{N} = 2$ super Yang-Mills theory in two dimensions [1]. The approach emphasizes the geometrical character of the twisted theory — the twist of $\mathcal{N} = 4$ that we consider contains only

[1] For examples of $\mathcal{N} = 1$ models with exact SUSY see the recent work [7] and [8]

integer spin fields and the fermion content is naturally embedded in a (real) Kähler-Dirac field. The connection between twisting and the Kähler-Dirac fermion mechanism has been emphasized in recent papers by Kawamoto et al.[14]. In this way a scalar supercharge $Q$ is produced and the bosonic field content of the model may be embedded in another Kähler-Dirac field. Using this field content we write down a $Q$-exact geometrical action which, after a simple change of variables, we show is nothing more but a well-known twist of $\mathcal{N}=4$ super Yang-Mills. We further show explicitly how to recover the conventional formulation involving spinor fields from the twisted action showing the complete equivalence of the twisted and untwisted theories.[2]

Finally, using a discretization prescription developed earlier, we are able to translate this geometrical theory to a hypercubic lattice while preserving gauge invariance and the twisted supersymmetry and *without* inducing any spurious zeroes in the spectrum of the lattice fermion operator. The price we pay for this is that the lattice theory requires a complexification of the degrees of freedom. We conjecture that we can restrict the path integrals needed to define the euclidean theory to the real line while preserving the Ward identities associated to the $Q$-symmetry.

## 2. Twisting and Kähler-Dirac fields

Consider the field content of $\mathcal{N}=4$ super Yang-Mills. It can be written in terms of 4 Majorana spinors $\Psi^i_\alpha$ where the index $\alpha$ labels the spinor degrees of freedom and the index $i=1\ldots4$ and is associated with an $\mathrm{SO}(4)_R$ R-symmetry of the fermionic action. The twisting procedure consists of constructing a new rotation group which is the diagonal subgroup of the original euclidean rotation group SO(4) and this R-symmetry [14 − 17].

$$\mathrm{SO}(4)' = \mathrm{diag}\left(\mathrm{SO}(4)\times \mathrm{SO}(4)_R\right). \tag{2.1}$$

This implies that the two indices $(\alpha, i)$ should be taken as equivalent and the fermion field is to be regarded as a matrix $\Psi^i_\alpha \to \Psi_{\beta\alpha}$. It is then natural to expand this matrix on a basis of products of $\gamma$-matrices

$$\Psi = \eta I + \psi_\mu\gamma_\mu + \frac{1}{2!}\chi_{\mu\nu}\gamma_\mu\gamma_\nu + \frac{1}{3!}\theta_{\mu\nu\lambda}\gamma_\mu\gamma_\nu\gamma_\lambda + \frac{1}{4!}\kappa_{\mu\nu\lambda\rho}\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\rho\,. \tag{2.2}$$

The coefficients $\eta$, $\psi_\mu$ etc are the *twisted* fields and clearly correspond to grassman valued tensors antisymmetric under the exchange of any two indices. There are exactly 16 independent fields in this decomposition and thus it is natural to take all these tensors (or p-forms) as real to match the 16 real supercharges of the original theory.

We can then assemble the component fields into a single so-called Kähler-Dirac field $\Psi = (\eta, \psi_\mu, \chi_{\mu\nu}, \theta_{\mu\nu\lambda}, \kappa_{\mu\nu\lambda\rho})$ It is then straightforward to show that solutions of the Dirac equation for four degenerate fermions (corresponding to the four columns of the matrix $\Psi$) can be gotten by solving the Kähler-Dirac equation [18 − 21]

$$\left(d - d^\dagger\right)\Psi = 0 \tag{2.3}$$

[2]In this paper we only consider theories in *flat* spacetime and there is thus no distinction between upper and lower indices for tensors

where the action of the exterior derivative operator $d$ on a $p$-form $\alpha$ yields a $(p+1)$-form $\beta$ whose components are given by

$$\beta_{\mu_1\ldots\mu_{p+1}} = \partial_{[\mu_{p+1}}\alpha_{\mu_1\ldots\mu_p]} \tag{2.4}$$

and the square bracket notation for the subscripts indicates an antisymmetrization with respect to all pairs of indices. The dot product of two such $p$-forms is defined by

$$\langle\alpha.\beta\rangle = \int dV \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p}\beta_{\mu_1\cdots\mu_p} \,. \tag{2.5}$$

With respect to this dot product we can then define the adjoint operator $-d^\dagger$ whose action on a $p$-form $\alpha$ yields a $(p-1)$-form $\beta$ with components

$$\beta_{\mu_2\ldots\mu_p} = \partial_{\mu_1}\alpha_{\mu_1\cdots\mu_p} \,. \tag{2.6}$$

These results also hold when the usual derivative is replaced by a gauge covariant derivative and all fields take values in the adjoint representation of some $\mathrm{U}(N)$ gauge group. It is also straightforward to verify that the Kähler-Dirac equation can be obtained from a Kähler-Dirac action of the form

$$S_{\mathrm{KD}} = \left\langle \Psi^\dagger.\left(d - d^\dagger\right)\Psi\right\rangle \tag{2.7}$$

or equivalently in matrix language

$$S_{\mathrm{KD}} = \frac{1}{2}\,\mathrm{Tr}\,\Psi^\dagger\gamma.D\Psi \,. \tag{2.8}$$

This representation of fermions in terms of $p$-forms is very natural from the lattice perspective as the latter may be associated with lattice $p$-cochains — functions defined on $p$-dimensional simplices in the lattice. Lattice analogs of the exterior derivative and its adjoint exist and allow us to discretize continuum actions formulated in geometric terms in a well-defined way. One of the most important consequences of such discretizations is that they prohibit spectrum doubling — the appearance of spurious zeroes of the fermion operator associated with lattice modes which do not appear in the continuum theory [19 – 21].

Notice that this twisting procedure will yield a scalar supercharge $Q$ which implies that the twisted theory will also contain a set of corresponding commuting p-form fields $\Phi = \left(\overline{\phi}, A_\mu, B_{\mu\nu}, W_{\mu\nu\lambda}, C_{\mu\nu\lambda\rho}\right)$. The fields $B_{\mu\nu}$ and $C_{\mu\nu\lambda\rho}$ will turn out to be multiplier fields which are integrated out of the final theory, leaving the bosonic fields to be represented by the gauge field $A_\mu$, the scalar $\overline{\phi}$, another scalar gauge parameter $\phi$ and the four independent degrees of freedom carried by $W_{\mu\nu\lambda}$. It is clear that in the twisting process the original 6 scalar fields have decomposed into a $\mathbf{4}+\mathbf{1}+\mathbf{1}$ of $\mathrm{SO}(4)'$. The appropriate action of $Q$ on these fields is a simple generalization of the two dimensional $\mathcal{N}=2$ case[3]

[3]Notice the minus sign which appears in the $Q$-variation of $\psi_\mu$ which was missing in our earlier paper [1] — many thanks to Mithat Unsal for pointing out this error

$$
\begin{aligned}
Q\overline{\phi} &= \eta \;\; Q\eta = \left[\phi, \overline{\phi}\right] \\
QA_\mu &= \psi_\mu \;\; Q\psi_\mu = -D_\mu \phi \\
QB_{\mu\nu} &= [\phi, \chi_{\mu\nu}] \;\; Q\chi_{\mu\nu} = B_{\mu\nu} \\
QW_{\mu\nu\lambda} &= \theta_{\mu\nu\lambda} \;\; Q\theta_{\mu\nu\lambda} = [\phi, W_{\mu\nu\lambda}] \\
QC_{\mu\nu\lambda\rho} &= [\phi, \kappa_{\mu\nu\lambda\rho}] \;\; Q\kappa_{\mu\nu\lambda\rho} = C_{\mu\nu\lambda\rho} \\
Q\phi &= 0\,. 
\end{aligned} \tag{2.9}
$$

Notice that, as expected from the twisted superalgebra [16], this charge is nilpotent up to a gauge transformation parametrized by the additional scalar field $\phi$ — $Q^2 = \delta^\phi_G$. Since the entire field content of this twisted model is given in terms of p-form fields it will be natural that only the exterior derivative and its adjoint may appear in the action of the twisted theory. This will guide us in the construction of the appropriate action.

## 3. Continuum action

### 3.1 Geometric formulation

We will hypothesize that the action can be written in a $Q$-exact form as for the two dimensional $\mathcal{N} = 2$ theory. Thus $S = \beta Q\Lambda$ where $\beta$ is a coupling and $\Lambda(\Psi, \Phi)$ will be termed a gauge fermion in agreement with the usual BRST terminology. All fields should be regarded as expanded on a basis of antihermitian generators of $\mathrm{U}(N)$. We choose $\Lambda$ to be of the form

$$
\begin{aligned}
\Lambda = \int d^4x \,\mathrm{Tr}\Bigg[&\chi_{\mu\nu}\left(F_{\mu\nu} + \frac{1}{2}B_{\mu\nu} - \frac{1}{2}\left[W_{\mu\lambda\rho}, W_{\nu\lambda\rho}\right] + D_\lambda W_{\lambda\mu\nu}\right) + \\
&+ \psi_\mu D_\mu \overline{\phi} + \frac{1}{4}\eta\left[\phi, \overline{\phi}\right] + \frac{1}{3!}\theta_{\mu\nu\lambda}\left[W_{\mu\nu\lambda}, \overline{\phi}\right] + \\
&+ \frac{1}{4!}\kappa_{\mu\nu\lambda\rho}\left(\sqrt{2}D_{[\mu}W_{\nu\lambda\rho]} + \frac{1}{2}C_{\mu\nu\lambda\rho}\right)\Bigg]\,. 
\end{aligned} \tag{3.1}
$$

Several of these terms are in common with the gauge fermion of $\mathcal{N} = 2$ super Yang-Mills theory in two dimensions. The new ones involve the 3 and 4-form fields. Of these the terms involving derivatives must be present to generate the correct Kähler-Dirac action for the twisted fermions (and will simultaneously generate the appropriate kinetic terms for the W-field). The commutator term involving $W_{\mu\nu\lambda}$ coupled to $\chi_{\mu\nu}$ will generate a quartic potential for the W-field analogous to that generated for the scalars $\phi$ and $\overline{\phi}$. This will allow contact to be made eventually with the supersymmetric theory where one expects the scalars and W-field to play similar roles. In the same way the commutator term involving $\overline{\phi}$ and $W$ will also generate the necessary mixed quartic couplings between the scalars and the W-field. Carrying out the $Q$-variation leads to the following action

$$
S = \beta\left(S_B + S_F + S_Y\right) \tag{3.2}
$$

where the piece of the action $S_B$ involving the bosonic fields takes the form

$$
\begin{aligned}
S_B = \int d^4x \,\mathrm{Tr}\Bigg[ & B_{\mu\nu}\left(F_{\mu\nu} - \frac{1}{2}\left[W_{\mu\lambda\rho}, W_{\nu\lambda\rho}\right] + D_\lambda W_{\lambda\mu\nu} + \frac{1}{2}B_{\mu\nu}\right) - \\
& - D_\mu\phi D_\mu\overline{\phi} + \frac{1}{4}[\phi,\overline{\phi}]^2 - \frac{1}{3!}\left[\phi, W_{\mu\nu\lambda}\right]\left[\overline{\phi}, W_{\mu\nu\lambda}\right] + \\
& + \frac{1}{4!}C_{\mu\nu\lambda\rho}\left(\sqrt{2}D_{[\mu}W_{\nu\lambda\rho]} + \frac{1}{2}C_{\mu\nu\lambda\rho}\right)\Bigg]
\end{aligned}
\tag{3.3}
$$

and the fermion kinetic terms are given by $S_F$ with

$$
S_F = \int d^4x \,\mathrm{Tr}\left[-\chi_{\mu\nu}D_{[\mu}\psi_{\nu]} - \chi_{\mu\nu}D_\lambda\theta_{\lambda\mu\nu} - \eta D_\mu\psi_\mu - \frac{\sqrt{2}}{4!}\kappa_{\mu\nu\lambda\rho}D_{[\mu}\theta_{\nu\lambda\rho]}\right]
\tag{3.4}
$$

and $S_Y$ contains the Yukawa couplings

$$
\begin{aligned}
S_Y = \int d^4x \,\mathrm{Tr}\Bigg[ & -\frac{1}{4}\eta\left[\phi,\eta\right] - \frac{1}{2}\frac{1}{4!}\kappa_{\mu\nu\lambda\rho}\left[\phi,\kappa_{\mu\nu\lambda\rho}\right] - \frac{1}{2}\chi_{\mu\nu}[\phi,\chi_{\mu\nu}] + \\
& + \psi_\mu\left[\overline{\phi},\psi_\mu\right] + \frac{1}{3!}\theta_{\mu\nu\lambda}\left[\overline{\phi},\theta_{\mu\nu\lambda}\right] + \\
& + \frac{1}{3!}\eta\left[\theta_{\mu\nu\lambda}, W_{\mu\nu\lambda}\right] - \frac{\sqrt{2}}{4!}\kappa_{\mu\nu\lambda\rho}\left[\psi_{[\mu}, W_{\nu\lambda\rho]}\right] + \\
& + \chi_{\mu\nu}\left[\theta_{\mu\lambda\rho}, W_{\nu\lambda\rho}\right] - \chi_{\mu\nu}\left[\psi_\lambda, W_{\lambda\mu\nu}\right]\Bigg].
\end{aligned}
\tag{3.5}
$$

Integrating over the multiplier fields $B_{\mu\nu}$ and $C_{\mu\nu\lambda\rho}$ and subsequently utilizing the Bianchi identity leads to a new bosonic action of the form

$$
\begin{aligned}
S_B = \int d^4x \,\mathrm{Tr}\Bigg[ & -\frac{1}{2}\left(\left(F_{\mu\nu} - \frac{1}{2}[W_{\mu\lambda\rho}, W_{\nu\lambda\rho}]\right)^2 + (D_\lambda W_{\lambda\mu\nu})^2 + \frac{2}{4!}\left(D_{[\mu}W_{\nu\lambda\rho]}\right)^2\right) - \\
& - D_\mu\phi D_\mu\overline{\phi} + \frac{1}{4}[\phi,\overline{\phi}]^2 - \frac{1}{3!}[\phi, W_{\mu\nu\lambda}][\overline{\phi}, W_{\mu\nu\lambda}]\Bigg].
\end{aligned}
\tag{3.6}
$$

### 3.2 Relation to the Marcus twist of $\mathcal{N} = 4$ SYM

At this point it is useful to trade the $W$, $\theta$, and $\kappa$ fields for new variables which will allow contact to be made between this theory and one of the conventional twists of $\mathcal{N} = 4$ super Yang-Mills. We write

$$
\begin{aligned}
W_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda\rho}V_\rho \\
\theta_{\mu\nu\lambda} &= \epsilon_{\mu\nu\lambda\rho}\overline{\psi}_\rho \\
\kappa_{\mu\nu\lambda\rho} &= \epsilon_{\mu\nu\lambda\rho}\overline{\eta}\,.
\end{aligned}
\tag{3.7}
$$

In terms of these variables the bosonic action reads

$$
\begin{aligned}
S_B = \int d^4x \,\mathrm{Tr}\Bigg[ & -\frac{1}{2}\left((F_{\mu\nu} - [V_\mu, V_\nu])^2 + \left(D_{[\mu}V_{\nu]}\right)^2 + 2\,(D_\mu V_\mu)^2\right) \\
& - D_\mu\phi D_\mu\overline{\phi} + \frac{1}{4}\left[\phi,\overline{\phi}\right]^2 - \left[\phi, V_\mu\right]\left[\overline{\phi}, V_\mu\right]\Bigg].
\end{aligned}
\tag{3.8}
$$

A further integration by parts yields a cancellation between the terms linear in $F_{\mu\nu}$ and the final bosonic action becomes

$$S_B = \int d^4x \, \mathrm{Tr} \left[ -\frac{1}{2}F_{\mu\nu}^2 - \frac{1}{2}[V_\mu, V_\nu]^2 - (D_\mu V_\nu)^2 - D_\mu \phi D_\mu \overline{\phi} + \frac{1}{4}[\phi, \overline{\phi}]^2 - [\phi, V_\mu][\overline{\phi}, V_\mu] \right]. \tag{3.9}$$

Making the additional rescalings $\chi \to 2\chi$ and $\overline{\eta} \to \frac{1}{\sqrt{2}}\overline{\eta}$ we find the fermion kinetic term takes the form

$$S_F = \int d^4x \, \mathrm{Tr} \left[ -2\chi_{\mu\nu} D_{[\mu} \psi_{\nu]} - 2\chi^*_{\mu\nu} D_{[\mu} \overline{\psi}_{\nu]} - 2\frac{\eta}{2} D_\mu \psi_\mu - 2\frac{\overline{\eta}}{2} D_\mu \overline{\psi}_\mu \right]. \tag{3.10}$$

where $\chi^*_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\chi_{\lambda\rho}$ is the dual field. In these variables the Yukawa's take on the more symmetrical form

$$\begin{aligned} S_Y = \int d^4x \, \mathrm{Tr} \Bigg[ &-\frac{\eta}{2}\left[\phi, \frac{\eta}{2}\right] - \frac{\overline{\eta}}{2}\left[\phi, \frac{\overline{\eta}}{2}\right] - 2\chi_{\mu\nu}\left[\phi, \chi_{\mu\nu}\right] + \psi_\mu \left[\overline{\phi}, \psi_\mu\right] + \overline{\psi}_\mu \left[\overline{\phi}, \overline{\psi}_\mu\right] + \\ &+ 2\frac{\eta}{2}\left[\overline{\psi}_\mu, V_\mu\right] - 2\frac{\overline{\eta}}{2}\left[\psi_\mu, V_\mu\right] + 4\chi_{\mu\nu}\left[\overline{\psi}_\mu, V_\nu\right] - 4\chi^*_{\mu\nu}\left[\psi_\mu, V_\nu\right] \Bigg]. \end{aligned} \tag{3.11}$$

The new action can be recognized as the twist of $\mathcal{N} = 4$ super Yang-Mills due to Marcus [22]. This is made explicit by the further change of variables

$$\begin{aligned} \eta_M &= \frac{1}{2}\left(\eta - i\overline{\eta}\right) \\ \psi_M &= \frac{1}{2}\left(\psi - i\overline{\psi}\right) \\ \chi_M &= 2\left(\chi - i\chi^*\right) \\ B_M &= \frac{1}{2}\phi \\ C_M &= \overline{\phi}\,. \end{aligned} \tag{3.12}$$

Notice that this twist of $\mathcal{N} = 4$ super Yang-Mills was also analyzed in [23] although in that paper two scalar supercharges were constructed which transformed into other under a duality operation. We will see later that such duality operations are incompatible with our latticization prescription and only the single supercharge corresponding to the transformations in equation 2.9 can be adapted to the lattice. It is interesting that this twist of $\mathcal{N} = 4$ can also be interpreted as a deformation of four dimensional super BF-theory [24].

### 3.3 Connection to conventional formulation of $\mathcal{N} = 4$ SYM

Finally we will show how this twisted model may be reinterpreted in terms of the usual formulation of $\mathcal{N} = 4$ super Yang-Mills theory. First, concentrate on the bosonic action and introduce the new fields ($\phi = \phi_1 + i\phi_2$)

$$\begin{aligned} X^\mu &= V_\mu \qquad \mu = 0 \ldots 3 \\ X^4 &= \phi_1 \\ X^5 &= \phi_2\,. \end{aligned} \tag{3.13}$$



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Then the bosonic action may be trivially rewritten as

$$S_B = -\frac{1}{2}F_{\mu\nu}^2 - \left(D_\mu X^i\right)^2 - \frac{1}{2}\sum_{ij}[X_i, X_j]^2 \,. \tag{3.14}$$

Notice this is real, positive semidefinite on account of the antihermitian basis for the fields. It is also precisely the bosonic sector of the $\mathcal{N} = 4$ super Yang-Mills action with $X^i$ the usual 6 real scalars of that theory.

Next let us turn our attention to the fermion kinetic term. From our previous discussion it should be clear that the twisted fermion kinetic term is nothing more than the component expansion of a Kähler-Dirac action:

$$S_F = \frac{1}{2}\,\mathrm{Tr}\,\Psi^\dagger \gamma.D\Psi \tag{3.15}$$

where $\Psi$ is the Kähler-Dirac field defined earlier with $\eta \to \eta/2$, $\theta_{\mu\nu\lambda} \to \epsilon_{\mu\nu\lambda\rho}\overline{\psi}_\rho$ and $\kappa_{\mu\nu\lambda\rho} \to \epsilon_{\mu\nu\lambda\rho}\overline{\eta}/2$. Naively such an action appears to describe a theory with *four* Dirac spinor fields - rather than the two one would have expected for $\mathcal{N} = 4$ super Yang-Mills. However it is evident that $\Psi$ obeys a reality condition if its associated Kähler-Dirac field is real. This reduces the action to that of two degenerate Dirac fermions. To see this in detail let us adopt a euclidean chiral basis for the $\gamma$-matrices

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}. \tag{3.16}$$

It is straightforward to see that the $\gamma$ matrices (and any products of them) obey a reality condition

$$\gamma_\mu^* = C\gamma_\mu C^{-1} \tag{3.17}$$

where the matrix $C$ is given by

$$C = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \tag{3.18}$$

With real p-form coefficients this implies a reality condition on $\Psi$ itself

$$\Psi^* = C\Psi C^{-1} \,. \tag{3.19}$$

This in turn implies that $\Psi^\dagger = C\Psi^T C^{-1}$ which makes it clear that the result of integrating over the Kähler-Dirac field $\Psi$ should be the pfaffian of the Kähler-Dirac operator (let us neglect the Yukawa couplings for the moment). This, in turn, will correspond to the product of *two* Dirac determinants.

To see this in more detail one can use the reality condition to show that successive columns $\Psi^{(n)}$ of $\Psi$ are not independent but are charge conjugates of each other

$$\Psi^{(2)} = C\left(\Psi^{(1)}\right)^* \qquad \Psi^{(4)} = C\left(\Psi^{(3)}\right)^*. \tag{3.20}$$

These conditions allow us to rewrite the twisted fermion kinetic term in the conventional form

$$\frac{1}{2}\sum_{\alpha=1,2} \lambda_\alpha^\dagger \gamma.D\lambda_\alpha \tag{3.21}$$

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where the spinors are read off as the first and third columns of the $\Psi$ matrix in this chiral basis:

$$\lambda_1 = \begin{pmatrix} \frac{\eta}{2} - \frac{\overline{\eta}}{2} + 2i\chi^+_{03} \\ -2\chi^+_{02} + 2i\chi^+_{01} \\ \left(\psi_0 + \overline{\psi}_0\right) + i\left(\psi_3 + \overline{\psi}_3\right) \\ -\left(\psi_2 + \overline{\psi}_2\right) + i\left(\psi_1 + \overline{\psi}_1\right) \end{pmatrix} \qquad \lambda_2 = \begin{pmatrix} \left(\psi_0 - \overline{\psi}_0\right) + i\left(-\psi_3 + \overline{\psi}_3\right) \\ \left(\psi_2 - \overline{\psi}_2\right) + i\left(\overline{\psi}_1 - \psi_1\right) \\ \frac{\eta}{2} + \frac{\overline{\eta}}{2} - 2i\chi^-_{03} \\ 2\chi^-_{02} - 2i\chi^-_{01} \end{pmatrix}. \tag{3.22}$$

Here, $\chi^\pm = \frac{1}{2}\left(\chi \pm \chi^*\right)$ are the usual self-dual and antiself-dual parts of the field. The Yukawa's can also be put in the general form

$$\frac{1}{2} \sum_{\alpha=1,2} \lambda^\dagger_\alpha C^\alpha_i \Gamma^i \left[X^i, \lambda_\alpha\right] \tag{3.23}$$

where $\Gamma^i = \{I, \gamma_5, \gamma_\mu\gamma_5, \mu = 0 \ldots 4\}$ and the coefficients $C^\alpha_i$ are just numbers. This is just the structure expected of $\mathcal{N} = 4$ super Yang-Mills. Notice that both Dirac operators $M^\alpha$ including the Yukawa interactions possess the symmetry

$$\left(D^\alpha\right)^* = C D^\alpha C^{-1} \tag{3.24}$$

which shows that their eigenvalves come in complex conjugate pairs and hence the associated determinants are positive definite.

Up to this point we have shown that the original $Q$-exact action written in terms of the component tensors of Kähler-Dirac fields 3.1 may be rewritten in terms of a conventional twisted action which may in turn be used to reconstruct *exactly* the usual formulations of $\mathcal{N} = 4$ super Yang-Mills theory. All this has been in the continuum. The recasting of the theory in terms of these geometrical fields is crucially important when devising a transcription to the lattice which preserves as much as possible of the continuum symmetry. We turn to this now.

## 4. Lattice action

### 4.1 Geometric formulation

The lattice action is obtained by discretization of the $Q$-exact action given in eq. 3.1. The prescription we employ was introduced in our earlier paper on $\mathcal{N} = 2$ super Yang-Mills in two dimensions and draws on the work in [25, 26]. It is summarized below

- A continuum p-form field $f_{\mu_1 \cdots \mu_p}(x)$ will be mapped to a corresponding lattice field associated with the $p$-dimensional hypercube at lattice site $x$ spanned by the (positively directed) unit vectors $\{\mu_1 \cdots \mu_p\}$.

- Such a lattice field is taken to transform under gauge transformations in the following way

$$f_{\mu_1 \cdots \mu_p}(x) \rightarrow G(x) f_{\mu_1 \cdots \mu_p}(x) G^{-1}(x + e_{\mu_1 \cdots \mu_p}) \tag{4.1}$$

  where the vector $e_{\mu_1 \cdots \mu_p} = \sum_{j=1}^{p} \mu_j$.

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- To construct gauge invariant quantities we will need to introduce both $f_{\mu_1\cdots\mu_p}$ and its hermitian conjugate $f^\dagger_{\mu_1\cdots\mu_p}(x)$. The latter transforms as

$$f^\dagger_{\mu_1\cdots\mu_p}(x) \to G(x+e_{\mu_1\cdots\mu_p})f_{\mu_1\cdots\mu_p}(x)G^{-1}(x)\,. \tag{4.2}$$

  Since for all fields bar the gauge field we will assume $f_{\mu_1\cdots\mu_p} = \sum_a f^a_{\mu_1\cdots\mu_p}T^a$ where $T^a$ are antihermitian generators of U($N$), this necessitates taking the fields $f^a_{\mu_1\cdots\mu_p}$ to be *complex*.

- For a continuum gauge field we introduce lattice link fields $U_\mu(x) = e^{A_\mu(x)} = e^{A^a_\mu(x)T^a}$ with $A^a_\mu(x)$ complex together with its conjugate $U^\dagger_\mu = e^{A^\dagger_\mu(x)}$.

- A covariant forward difference operator can be defined which acts on a field $f_{\mu_1\cdots\mu_p}(x)$ as follows

$$D^+_\mu f_{\mu_1\cdots\mu_p}(x) = U_\mu(x)f_{\mu_1\cdots\mu_p}(x+\mu) - f_{\mu_1\cdots\mu_p}(x)U_\mu(x+e_{\mu_1\cdots\mu_p})\,. \tag{4.3}$$

  This operator acts like a lattice exterior derivative with respect to gauge transformations in mapping a $p$-form lattice field to a $(p+1)$-form lattice field.

- Similarly we can define an adjoint operator $D^-_\mu$ whose action on some field $f_{\mu_1\cdots\mu_p}$ is given by

$$D^-_\mu f_{\mu_1\cdots\mu_p}(x) = f_{\mu_1\cdots\mu_p}(x)U^\dagger_\mu(x+e_{\mu_1\cdots\mu_p}-\mu) - U^\dagger_\mu(x-\mu)f_{\mu_1\cdots\mu_p}(x-\mu)\,. \tag{4.4}$$

  It thus acts like the adjoint of the exterior derivative.

- Following [19] all instances of $\partial_\mu$ in the continuum action will be replaced by $D^+_\mu$ if the derivative acts like $d$ (curl-like operation) and $D^-_\mu$ if the derivative acts like $d^\dagger$ (divergence-like operation). One can show using results from homology theory [19] that this guarantees that the lattice theory will exhibit no spectrum doubling.

Notice that each $p$-hypercube (for $p>0$) possesses two orientations and this gives one natural explanation of the doubling of degrees of freedom exhibited by the lattice theory[4]. The need for this doubling can be easily seen at an operational level — with the lattice gauge transformation rules we have given the lattice operator

$$\int d^4x A_{\mu_1\ldots\mu_d}B_{\mu_1\ldots\mu_d} \tag{4.5}$$

is *not* gauge invariant (and if used in the gauge fermion will also lead to a violation of $Q$-invariance). However, there is a natural way to construct a gauge invariant lattice operator

$$\int d^4x A^\dagger_{\mu_1\ldots\mu_d}B_{\mu_1\ldots\mu_d}\,. \tag{4.6}$$

[4]Another possible interpretation relates to the existence of both lattice and dual lattice

However, the differing gauge transformations of $A^{\dagger}_{\mu_1\ldots\mu_d}$ and $A_{\mu_1\ldots\mu_d}$ then require that the component fields $A^a_{\mu_1\ldots\mu_d}$ be taken as complex. Clearly this prescription must be used consistently for all fields and necessitates treating the gauge links as *non-unitary* matrices. This rule naturally leads to a gauge action of the form $F^{\dagger}_{\mu\nu}F_{\mu\nu}$ which, we will see later, reduces to the usual Wilson action in the unitary limit — a feature which would not have occurred with the lattice operator $F_{\mu\nu}F_{\mu\nu}$.

Notice that our gauge transformation rules allow for a complex gauge transformation parameter $\phi(x)$. This is natural in such a complexified theory. Compatibility with the dagger operation requires that $\phi^{\dagger} = -\phi$ which should be true for *all* scalar fields. While the symmetries are most easily implemented in the complexified theory we will argue that the final path integral can be restricted to the real line without violating either gauge invariance or the twisted supersymmetry.

Using these ingredients we can easily transfer the continuum gauge fermion in eq. 3.1 to the lattice obtaining

$$\begin{aligned}\Lambda = \frac{1}{2}\sum_x \mathrm{Tr}\Big[ &-\chi^{\dagger}_{\mu\nu}\left(F_{\mu\nu} + \frac{1}{2}B_{\mu\nu} - \frac{1}{2}\left[W_{\mu\lambda\rho}, W_{\nu\lambda\rho}\right]' + D^{-}_{\lambda}W_{\lambda\mu\nu}\right) - \\ &-\psi^{\dagger}_{\mu}D^{+}_{\mu}\overline{\phi} - \frac{1}{4}\eta^{\dagger}\left[\phi,\overline{\phi}\right] - \frac{1}{3!}\theta^{\dagger}_{\mu\nu\lambda}\left[W_{\mu\nu\lambda},\overline{\phi}\right] - \\ &-\frac{1}{4!}\kappa^{\dagger}_{\mu\nu\lambda\rho}\left(\sqrt{2}D^{+}_{[\mu}W_{\nu\lambda\rho]} + \frac{1}{2}C_{\mu\nu\lambda\rho}\right) + \mathrm{h.c}\Big]\end{aligned} \tag{4.7}$$

Notice that it is necessary to add the hermitian conjugates of these terms to the gauge fermion to obtain a real lattice action. The extra minus signs just reflect the antihermitian nature of our basis matrices $T^a$. Unlike the continuum theory we are no longer at liberty to transform 3-form and 4-form fields to their duals since this would break lattice gauge invariance. The lattice field strength $F_{\mu\nu}$ is obtained from the simple relation

$$F_{\mu\nu}(x) = D^{+}_{\mu}U_{\nu}(x)\,. \tag{4.8}$$

Notice that it automatically antisymmetric in its indices and reduces to the usual Yang-Mills field strength in the naive continuum limit. The prime on the commutator of two $W$-fields indicates that this term must be modified in the lattice theory in order to maintain gauge invariance. The following definition of the commutator yields a term which transforms as a lattice 2-form

$$\left\{W_{\mu\lambda\rho}(x)W_{\nu\lambda\rho}(x+\mu+\lambda+\rho) - W_{\nu\lambda\rho}(x)W_{\mu\lambda\rho}(x+\nu+\lambda+\rho)\right\}P(x+\mu+\nu;\lambda,\rho) \tag{4.9}$$

where the ordered link path $P(x+\mu+\nu;\lambda,\rho)$ is necessary to ensure that the gauge transformation properties of the commutator do not depend on the indices $\lambda$ and $\rho$.

$$P(x;\lambda,\rho) = U^{\dagger}_{\lambda}(x+\lambda+2\rho)U^{\dagger}_{\lambda}(x+2\rho)U^{\dagger}_{\rho}(x+\rho)U^{\dagger}_{\rho}(x)\,. \tag{4.10}$$

To enforce lattice rotational symmetry one can replace $P(x;\lambda,\rho)$ by a symmetrized average over all lattice paths leading from $(x+2\lambda+2\rho)$ to $x$. Notice that this lattice commutator term reduces to the continuum commutator in the naive continuum limit in which the gauge links are set to unity. Actually there is one further wrinkle to comment on. The commutator term involving $W$ and $\overline{\phi}$ must also be modified from its naive continuum form to maintain gauge invariance. Since this same procedure must be used in the lattice $Q$-transformations we list the general rule for the commutator of a scalar field with an arbitrary lattice $p$-form

$$[\phi, f_{\mu_1\cdots\mu_p}] = \phi(x) f_{\mu_1\cdots\mu_p}(x) - f_{\mu_1\cdots\mu_p}(x)\phi(x + e_{\mu_1\cdots\mu_p})\,. \tag{4.11}$$

The $Q$-transformations listed in eq. 2.9 can now be taken over almost trivially to the lattice. They take the form

$$\begin{aligned}
Q\overline{\phi} &= \eta \quad Q\eta = [\phi, \overline{\phi}] \\
QU_\mu &= \psi_\mu \quad Q\psi_\mu = -D^+_\mu \phi \\
QB_{\mu\nu} &= [\phi, \chi_{\mu\nu}] \quad Q\chi_{\mu\nu} = B_{\mu\nu} \\
QW_{\mu\nu\lambda} &= \theta_{\mu\nu\lambda} \quad Q\theta_{\mu\nu\lambda} = [\phi, W_{\mu\nu\lambda}] \\
QC_{\mu\nu\lambda\rho} &= [\phi, \kappa_{\mu\nu\lambda\rho}] \quad Q\kappa_{\mu\nu\lambda\rho} = C_{\mu\nu\lambda\rho} \\
Q\phi &= 0
\end{aligned} \tag{4.12}$$

where gauge invariance dictates the use of a forward difference in the variation of $\psi_\mu$ and the commutators are to be point split in the way described above so that they can be interpreted as infinitessimal gauge transformations on lattice $p$-form fields. For example,

$$[\phi, \chi_{\mu\nu}] \to \phi(x)\chi_{\mu\nu}(x) - \chi_{\mu\nu}(x)\phi(x + \mu + \nu)\,. \tag{4.13}$$

The $Q$-transformations of the daggered fields can be found by taking the hermitian conjugate of these transformations together with the requirement that scalar fields transform into (minus) themselves under the dagger operation.

Carrying out the $Q$-variation and subsequently integrating out the multiplier fields as for the continuum case leads to the following components of the lattice action

$$\begin{aligned}
S_B = \frac{1}{2}\sum_x \mathrm{Tr}\Bigg[&\left(F_{\mu\nu} - \frac{1}{2}[W_{\mu\lambda\rho}, W_{\nu\lambda\rho}]' + D^-_\lambda W_{\lambda\mu\nu}\right)\left(F_{\mu\nu} - \frac{1}{2}[W_{\mu\lambda\rho}, W_{\nu\lambda\rho}]' + D^-_\lambda W_{\lambda\mu\nu}\right)^\dagger \\
&+ \frac{2}{4!}\left(D^+_{[\mu} W_{\nu\lambda\rho]}\right)\left(D^+_{[\mu} W_{\nu\lambda\rho]}\right)^\dagger + \left(D^+_\mu \phi\right)^\dagger \left(D^+_\mu \overline{\phi}\right) + \frac{1}{4}[\phi, \overline{\phi}]^2 \\
&- \frac{1}{3!}[\phi, W^\dagger_{\mu\nu\lambda}][W_{\mu\nu\lambda}, \overline{\phi}] + \mathrm{h.c}\Bigg].
\end{aligned} \tag{4.14}$$

Notice that the terms listed above are positive semidefinite along the contour $\overline{\phi} = -\phi^\dagger$ corresponding to choosing $\overline{\phi}^a(x) = (\phi^a(x))^*$. Notice also that the lattice Yang-Mills action $F^\dagger_{\mu\nu}F_{\mu\nu}$ takes the form

$$\mathrm{Tr}\sum_x \sum_{\mu<\nu}\left(2I - U^P_{\mu\nu} - (U^P_{\mu\nu})^\dagger\right) + \mathrm{Tr}\sum_x \sum_{\mu<\nu}\left(M_{\mu\nu} + M_{\nu\mu} - 2I\right) \tag{4.15}$$

where

$$U^P_{\mu\nu} = \mathrm{Tr}\left(U_\mu(x)U_\nu(x+\mu)U^\dagger_\mu(x+\nu)U^\dagger_\nu(x)\right) \tag{4.16}$$

is the usual Wilson plaquette operator and

$$M_{\mu\nu}(x) = U_\mu(x)U^\dagger_\mu(x)U_\nu(x+\mu)U^\dagger_\nu(x+\mu)\,. \tag{4.17}$$

Notice that the latter term vanishes if we restrict the gauge links to be unitary. Turning now to the fermion kinetic term we find

$$\begin{aligned} S_F = \sum_x \mathrm{Tr}\Bigg[&\frac{1}{2!}\chi^\dagger_{\mu\nu}D^+_{[\mu}\psi_{\nu]} + \psi^\dagger_\nu D^-_\mu\chi_{\mu\nu} + \frac{1}{2!}\chi^\dagger_{\mu\nu}D^-_\lambda\theta_{\lambda\mu\nu} + \frac{1}{3!}\theta^\dagger_{\mu\nu\lambda}D^+_{[\mu}\chi_{\nu\lambda]} + \\ &+ \psi^\dagger_\mu D^+_\mu\frac{\eta}{2} + \frac{\eta^\dagger}{2}D^-_\mu\psi_\mu + \frac{1}{4!}\frac{\kappa_{\mu\nu\lambda\rho}}{2}D^+_{[\mu}\theta_{\nu\lambda\rho]} + \frac{1}{3!}\theta^\dagger_{\mu\nu\lambda}D^-_\mu\frac{\kappa_{\mu\nu\lambda\rho}}{2}\Bigg]. \end{aligned} \tag{4.18}$$

In this expression we have rescaled $\kappa_{\mu\nu\lambda\rho} \to \frac{1}{\sqrt{2}}\kappa_{\mu\nu\lambda\rho}$ as in the continuum. The Yukawa couplings follow in a similar manner

$$\begin{aligned} S_Y = \frac{1}{2}\sum_x \mathrm{Tr}\Bigg[&\frac{\eta^\dagger}{2}\left[\phi, \frac{\eta}{2}\right] + \frac{1}{4!}\frac{\kappa^\dagger_{\mu\nu\lambda\rho}}{2}\left[\phi, \frac{\kappa_{\mu\nu\lambda\rho}}{2}\right] + \frac{1}{2}\chi^\dagger_{\mu\nu}\left[\phi, \chi_{\mu\nu}\right] - \\ &- \psi^\dagger_\mu\left[\overline{\phi}, \psi_\mu\right] - \frac{1}{3!}\theta^\dagger_{\mu\nu\lambda}\left[\overline{\phi}, \theta_{\mu\nu\lambda}\right] + \\ &+ 2\frac{1}{3!}\theta^\dagger_{\mu\nu\lambda}\left[W_{\mu\nu\lambda}, \frac{\eta}{2}\right] + 2\frac{1}{4!}\frac{\kappa^\dagger_{\mu\nu\lambda\rho}}{2}\left[\psi_{[\mu}, W_{\nu\lambda\rho]}\right] - \\ &- \frac{1}{2}\chi^\dagger_{\mu\nu}\left[\theta_{\lambda\rho[\mu}, W_{\nu]\lambda\rho}\right]' + \chi^\dagger_{\mu\nu}\left[\psi^\dagger_\lambda, W_{\lambda\mu\nu}\right] + \mathrm{h.c}\Bigg]. \end{aligned} \tag{4.19}$$

The commutator term involving $\psi_\mu$ and $W_{\nu\lambda\rho}$ is easily found to take the gauge covariant and point split form

$$\left(\psi_{[\mu}(x)W_{\nu\lambda\rho]}(x+\mu) - W_{[\nu\lambda\rho}(x)\psi_{\mu]}(x+\nu+\lambda+\rho)\right). \tag{4.20}$$

Similarly that involving $\psi^\dagger_\lambda$ and $W_{\lambda\mu\nu}$ is found to be

$$\Big(W_{\lambda\mu\nu}(x)\psi^\dagger_\lambda(x+\mu+\nu) - \psi^\dagger_\lambda(x-\lambda)W_{\lambda\mu\nu}(x-\lambda)\Big)\,. \tag{4.21}$$

There is one further term which arises from the $Q$-variation of the $P(x;\lambda,\rho)$ factor in the primed commutator. This yields

$$-\frac{1}{2}\chi^\dagger_{\mu\nu}[W_{\mu\lambda\rho}, W_{\nu\lambda\rho}]Q\left(P(x+\mu+\nu;\lambda\rho)\right). \tag{4.22}$$

This operator will vanish in the naive continuum limit as expected.

Consider now the twisted lattice fermion action $S_F + S_Y$. In the continuum this can be recast in terms of a single Kähler-Dirac field $\Psi$.

$$S_{F+Y} = \Psi^\dagger M(U,\phi,V)\Psi\,. \tag{4.23}$$

On the lattice, it is convenient, although *not essential* to to rewrite it in terms of a single *complex* Kähler-Dirac field $\Psi$. However the following Yukawa coupling is problematic in this respect as it involves a $\Psi^\dagger\Psi^\dagger$ coupling

$$\chi^\dagger_{\mu\nu}(x)\left[\psi^\dagger_\lambda(x), W_{\lambda\mu\nu}(x)\right]. \tag{4.24}$$

Such a term could be accommodated in the lattice theory by doubling the number of fermion fields (and taking an appropriate further square root of the fermion determinant when calculating the effective bosonic action). Such an approach has the merit of preserving the $Q$-symmetry exactly. Alternatively, one can get around this problem by replacing this term with the another similar one

$$\chi^{\dagger}_{\mu\nu}\left[U^{\dagger}_{\lambda}(x)\psi_{\lambda}(x)U^{\dagger}_{\lambda}(x), W_{\lambda\mu\nu}(x)\right]. \tag{4.25}$$

This modification of the lattice theory corresponds to a soft breaking of the twisted SUSY and as such should not lead to dangerous SUSY violating corrections. The additional soft breaking terms would then vanish in the continuum limit. Notice that a $Q$-violating gluino mass term is likely to be necessary to control any numerical algorithm used to simulate this theory and such a term will look rather like the modified Yukawa we discuss here. However, further work will be needed to completely clarify this issue, in particular, how the presence of such a soft breaking term effects the other twisted supersymmetries.

Finally, using this latter prescription leads, after integration over the complex twisted fermions, to a factor of $\det M(U,\phi,V)$ in the effective action for the bosonic fields.

### 4.2 Continuum limit

We have formulated the lattice theory in terms of a set of complex fields since in this way we can preserve both gauge invariance and the twisted supersymmetry. However to target $\mathcal{N}=4$ super Yang-Mills we would like to restrict the path integrals defining correlation functions to the real line (actually, in the case of the scalars we need to impose the condition $\overline{\phi}=-\phi^{\dagger}$). The first point to note is that the on-shell bosonic action $S_B$ is still gauge invariant if the imaginary parts of $W_{\mu\nu\lambda}$, $U_\mu$ are set to zero. Furthermore, as in the continuum theory, we will assume that the correct weight for the lattice twisted fermions after projection to the real line is just the pfaffian of the full Kähler-Dirac operator (including the Yukawas). The latter can be replaced by $\det^{\frac{1}{2}}M(U,\phi,V)$ which is then also gauge invariant. Finally, it should be clear that the lattice action we have proposed reduces in the naive continuum limit to the usual super Yang-Mills form along the real line. So the remaining question is whether the Ward identities corresponding to the twisted supersymmetry $Q$ continue to hold along this contour. We have no proof of this statement but can make the following plausibility argument.

The lattice action is $Q$-exact which implies that any $Q$-invariant observable can be computed in the limit $\beta\to\infty$. This limit corresponds to the naive continuum limit in which the gauge links are expanded to first order in the complex gauge field $C_\mu=A_\mu+iB_\mu$ and lattice expressions can be identified with their continuum counterparts. Consider the (complexified) Yang-Mills field strength that arises in such a limit

$$\beta\left[\left(F_{\mu\nu}(A)-[B_\mu,B_\nu]\right)^2+\left(D_{[\mu}B_{\nu]}\right)^2\right]. \tag{4.26}$$

As $\beta\to\infty$ it is clear that the path integral is saturated by fields with $B_\mu=0$ and $F_{\mu\nu}=0$. Thus it should be possible to compute any $Q$-invariant observable on the subspace corresponding to $B_\mu=0$. The Ward identities are generated by picking a trivial $Q$-invariant observable, namely $QO(\Psi, \Phi)$. We can thus conjecture that these Ward identities will hold *on the real line* at finite lattice spacing. This conjecture should be checked by perturbative calculations and numerical simulation.

## 5. Conclusions

In this paper we propose a discretization of the $\mathcal{N} = 4$ twisted Yang-Mills action in four dimensions which is a generalization of the procedure used earlier to construct a lattice theory of $\mathcal{N} = 2$ super Yang-Mills theory in two dimensions [1]. The approach emphasizes the geometrical character of the twisted theory — the twist of $\mathcal{N} = 4$ that we consider contains only integer spin fields and the fermion content is naturally embedded in a (real) Kähler-Dirac field with anticommuting components. This decomposition of the fermions also implies the existence of a scalar supercharge $Q$ which is nilpotent up to gauge transformations. The superpartners of the fermions are now contained in another Kähler-Dirac field. We write down a simple set of transformation rules for the fields under this supercharge. A $Q$-exact action is derived and shown to reduce to a well-known twisting of $\mathcal{N} = 4$ super Yang-Mills after a suitable change of variables. For completeness we also show how the original spinor formulation of the theory can be recovered from this twisted model.

This manifestly geometric starting point allows us to discretize the theory without inducing spectrum doubling and maintaining both gauge invariance and a single twisted supersymmetry. The lattice theory naturally contains complex fields — to access the correct continuum limit requires that an appropriate contour be chosen when evaluating the path integral. We argue that both gauge invariance and the twisted supersymmetry can be maintained if this contour is chosen such that the imaginary parts of all component fields bar the scalars are taken to vanish. The scalars $\phi^a(x)$ and $\overline{\phi}^a(x)$ are taken to be complex conjugates of each other. The resulting effective action is real, positive semidefinite at least for small enough lattice spacing.

There are several directions for further work. The most obvious is the need for both perturbative and numerical checks on the twisted supersymmetric Ward identities corresponding to $Q$. If the results of those are positive it will be necessary to derive and examine the Ward identities following from additional elements of the twisted superalgebra. In general we would expect that these latter Ward identities would be broken at finite lattice spacing. The hope is that the presence of a single exact supersymmetry will, however, largely prohibit the theory from developing relevant operators breaking these additional supersymmetries. In this context it will be crucial to pursue further studies, both analytical and numerical, directed at allowing us to understand what, if any, additional fine tuning is needed to reach the full $\mathcal{N} = 4$ theory in the continuum limit.

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